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Another Solution to the Schrödinger-Langevin Equation[1]

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Abstract

Introduction: an alternative solution to the Schrodinger-Langevin equation is presented, where the temporal dependence is explained, assuming a Coulomb potential. Finally, the trajectory equations are found. **Objective**: in this paper we contribute by presenting a detailed and simple solution of the Schrödinger-Langevin equation for a Coulomb potential. **Materials and Methods**: using an appropriate ansatz, we solve the Schrödinger-Langevin equation, finding the expected values of position and moment. **Results:** a simple method was presented to find the expected position and moment values in the Schrödinger-Langevin equation, the ansatz used to find these solutions allows the model to be generalized in a certain way to electric potentials and harmonic oscillators. **Conclusions:** the model used to solve the Schrödinger-Langevin equation, allowed to find the expected values of position and moment of a particle in a Coulomb potential, the temporal dependence of such solutions is made explicit, which allows finding the path equations of the particles.

Keywords: Schrödinger-Langevin equation, quantum friction, interpretation of Bhom, expected value of the position.

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Una Solución Alterna a la Ecuación de Schrödinger-Langevin

Resumen

Introducción: se presenta una solución alternativa a la ecuación de Schrodinger-Langevin, donde se explica la dependencia temporal, asumiendo un potencial de Coulomb. Finalmente, se encuentran las ecuaciones de trayectoria. **Objetivo:** en este trabajo hacemos una contribución presentando una solución detallada y sencilla de la ecuación de Schrödinger-Langevin para un potencial de Coulomb. **Materiales y Métodos:** usando un ansatz apropiado, solucionamos la ecuación de Schrödinger-Langevin, encontrando los valores esperados de posición y momento. **Resultados:** se presentó un método sencillo para hallar los valores esperados de posición y momento en la ecuación de Schrödinger-Langevin, el ansatz utilizado para encontrar estas soluciones permite generalizar en cierta forma el modelo a potenciales eléctricos y osciladores armónicos. **Conclusiones:** el modelo utilizado para solucionar la ecuación de Schrödinger-Langevin, permitió encontrar los valores esperados de posición y momento de una partícula en un potencial de Coulomb, se explicita la dependencia temporal de tales soluciones lo que permite encontrar las ecuaciones de trayectoria de las partículas.

Palabras clave: Ecuación de Schrödinger-Langevin, fricción cuántica, interpretación de Bhom, valor esperado de la posición.

Uma Solução Alternativa para a Equação Schrödinger-Langevin

Resumo

Introdução: uma solução alternativa para a equação de Schrodinger-Langevin é apresentada, onde a dependência temporal é explicada, assumindo um potencial de Coulomb. Finalmente, existem as equações de caminho **Objetivo:** neste trabalho fazemos uma contribuição apresentando uma solução simples e detalhada da equação de Schrödinger-Langevin para um potencial de Coulomb. **Materiais e métodos:** usando um ansatz apropriado, resolvemos a equação de Schrödinger-Langevin, encontrando os valores esperados de posição e momento. **Resultados:** foi apresentado um método simples para encontrar os valores esperados de posição e momento na equação de Schrödinger-Langevin, o ansatz utilizado para encontrar essas soluções permite que o modelo seja generalizado de certa forma para potenciais elétricos e osciladores harmônicos. **Conclusões:** o modelo utilizado para resolver a equação de Schrödinger-Langevin, permitiu encontrar os valores esperados de posição e momento de uma partícula em um potencial de Coulomb, sendo explicitada a dependência temporal de tais soluções, o que permite encontrar as equações de caminho das partículas.

Palavras-chave: Equação de Schrödinger-Langevin, atrito quântico, interpretação de Bhom, valor esperado da posição.

Introduction

The macroscopic objects obey Newton's equations of motion , where is the position of the particle in time and is the force acting on the particle. By giving values for both position and velocity in a time it is possible to calculate the trajectory of an object and as a result it is possible to predict the position and velocity at a later time.

At the molecular and atomic level objects obey the laws of quantum mechanics, in this way Newton's laws are not strictly valid, the unique concept of trajectory is no longer literally valid. This is the fundamental basis of quantum mechanics, which is not local, each component of the system has influence on the movement of any other part which is reciprocal.

In this paper we take the Schrödinger-Langevin equation. To study processes of diffusion and dissipation at the quantum level, we generally start from the Schrödinger-Langevin equation in the study of the Brownian movement (Kostin 1972), which contains two new terms, which include a friction constant and the other one ends a random potential. Taking a Coulomb potential and find a possible solution to the Schrödinger-Langevin equation, comparing the interpretations according to quantum mechanics and the casual interpretation of Bhom (Bohm D, 1996) by quantum trajectories.

The article is organized as follows; in the first part the terms of the Schrödinger-Langevin equation are presented, in the second part we implement the proposed solution, as a linear exponential function in , in addition we find the equations of movement, and finally a discussion of this result is made.

Materials and Methods

The Schrödinger-Langevin equation

In quantum mechanics any physical system can be associated with a Hermitic operator $\widehat{H}$, which determines the temporal evolution of the system according to:

$$i\frac{\partial\psi}{\partial t} = \widehat{H}\,\psi\,.$$

In the Langevin equation (Chia-ChunChou, 2017) of motion, the Hamiltonian operator includes the expression of the potential V_L associated with quantum friction processes; the Schrödinger equation in one dimension is given by (Kostin 1972):

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x,t) + V_R\,\psi(x,t) + V_L\,\psi(x,t), \qquad (1)$$

where (Kostin 1972):

$$V_L = \frac{\hbar\Gamma}{2i\,m}\,ln\left[\frac{\psi(x,t)}{\psi^*(x,t)}\right] + W(t)$$

V_R is the potential associated to the random force (in this work it is taken as null), the term $W(t)$ that originates from the potential associated with friction, can be removed by introducing the transformation:

$$\psi(x,t) = e^{i\theta(t)}\,\phi(x,t), \qquad (2)$$

making the corresponding substitutions, you get to:

$$i\,\hbar\,\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + V(x)\phi(x,t) \ + \frac{\hbar\Gamma}{2im}\,\phi(x,t)\,ln\left[\frac{\phi(x,t)}{\phi^{*}(x,t)}\right], \qquad (3)$$

also

$$\hbar\theta^{'}(t) + \ \hbar\Gamma\,\theta(t) = -\,W(t),$$

The equation (3) is the one we use to study the behavior of a particle of mass m in a Coulomb potential.

Solution assuming Coulomb potential

Assuming a Coulomb potential $V = -qEx = -\,kx;\ \ k = qE$, the equation (3) takes the form:

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} - kx\,\phi(x,t) \ + \frac{\hbar\Gamma}{2i\,m}\,\phi(x,t)\,ln\left[\frac{\phi(x,t)}{\phi^{*}(x,t)}\right], \qquad (4)$$

Assuming as solution of the equation (4) of the form:

$$\phi(x,t) = \ A(x,t)\,e^{i(\alpha_1(t)x+\alpha_2(t))} \qquad (5)$$

performing the first time derivative

$$\frac{\partial\phi}{\partial t} = e^{i(\alpha_1 x+\alpha_2)}\left(A' \ + iA(\alpha'_1\,x \ + \alpha'_2)\right),$$

of the second derivative with respect to time, we obtain

$$\frac{\partial^2\phi}{\partial x^2} = e^{i(\alpha_1 x+\alpha_2)}\left(\frac{\partial^2 A}{\partial x^2} + \ 2\,i\,\alpha_1\,\frac{\partial A}{\partial x} - \ \alpha_1^2 A\right),$$

Equating the imaginary and real parts, the system of equations is obtained:

$$A' = -\frac{\hbar}{m}\,\alpha_1\,\frac{\partial A}{\partial x} \qquad (6)$$

$$-\hbar A\,(\alpha'_1 x + \alpha'_2) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 A}{\partial x^2} - \alpha_1^2 A\right) - kxA + \frac{\hbar\Gamma}{m}A(\alpha_1 x + \alpha_2) \qquad (7)$$

Assuming the function $A(x;t) = G(x)F(t)$ (separable variables); of the equation (6)

$$\frac{F'(t)}{\alpha_1 F} = -\frac{\hbar}{Gm}\frac{\partial G}{\partial x} = a$$

Where a is constant. Then:

$$G(x) = Ce^{-\frac{amx}{\hbar}} \qquad (8)$$

How $A(x;t) = G(x)F(t)$ and considering that

$$\frac{1}{G}\frac{\partial^2 G}{\partial x^2} = \frac{m^2 a^2}{\hbar^2},$$

substituting in (7) we obtain:

$$-\hbar\,(\alpha'_1 x + \alpha'_2) = -\frac{\hbar^2}{2m}\left(\frac{m^2 a^2}{\hbar^2} - \alpha_1^2\right) - k\,x + \frac{\hbar\Gamma}{m}(\alpha_1 x + \alpha_2), \qquad (9)$$

Equation of coefficients in x on both sides that leads to:

$$-\hbar\;\alpha_1'(t) = -k + \frac{\hbar\Gamma}{m}\;\alpha_1(t)$$

This is a first order differential equation for a (1), it is found that

$$-\alpha_1'(t) = -\frac{k}{\hbar} + \frac{\Gamma}{m}\alpha_1(t); \qquad \alpha_1(t) = be^{-\frac{\Gamma t}{m}} + \frac{km}{\hbar\,\Gamma}, \qquad (10)$$

Independent terms:

$$\alpha_2' + \frac{\Gamma}{m}\alpha_2 = \frac{a^2 m}{2\hbar} - \frac{\hbar}{2m}\left(be^{-\frac{\Gamma t}{m}} + \frac{k\,m}{\hbar\Gamma}\right)^2$$

Integrating you get

$$\alpha_2(t) = \frac{a^2m^2}{2\hbar\,\Gamma^2} - \frac{k^2\,m^2}{2\,\hbar\,\Gamma^3} + \frac{b^2\hbar}{2\,\Gamma}\;e^{-\frac{2\,\Gamma t}{m}} - \frac{bk}{\Gamma}t\,e^{-\frac{\Gamma t}{m}} + d\,e^{-\left(\frac{\Gamma t}{m}\right)}, \qquad (11)$$

d constant; Solving for $F(t)$

$$\frac{F'}{F} = a\alpha_1 = a\;\left(b\,e^{-\frac{\Gamma t}{m}} + \frac{km}{\hbar\Gamma}\right),$$

$$F(t) = \;C\,e^{\frac{am}{\Gamma}\left(\frac{kt}{\hbar} - b\,e^{-\Gamma t/m}\right)}\;, \qquad (12)$$

where C is constant. Then the A function takes the form:

$$A(x,t) = Ce^{-\frac{amx}{\hbar}}\;e^{\frac{a\,m}{\Gamma}\left(\frac{k\,t}{\hbar} - b\,e^{-\Gamma t/m}\right)}\;, \qquad (13)$$

Taking the equation (2), (5), and (13). You finally get the function $\psi(x,t)$

$$\psi(x,t) = Ce^{i\theta(t)}\;e^{-\frac{amx}{\hbar}}\;e^{\frac{a\,m}{\Gamma}\left(\frac{k\,t}{\hbar} - b\,e^{-\Gamma\,t/m}\right)}\;e^{i(\alpha_1\,(t)\,x + \alpha_2\,(t))}, \qquad (14)$$

With $\alpha_1(t), \alpha_2(t)$ given by the equations (10) and (11). Assuming a suggested interpretation of quantum theory in terms of hidden variables, type Bohm (Bohm D. y B. J. &. B. J. S. Hiley, 1996. Bohm D. 1952).

$$\psi(x,t) = R(x,t)e^{IS(x,t)\,/\hbar}, \qquad (15)$$

Where $R(x;\;t);\;S(x;\;t)$ are real functions, then we have:

$$S(x,t) = \frac{\hbar}{2i}\,ln\left[\frac{\psi}{\psi^*}\right]\;, \qquad (16)$$

taking the equation found in (14) and (16) substituting in (4) you get:

$$S(x,t)\;\; = \hbar\;(\;\theta(t) + \alpha_1\;(t)\;x\;\; + \alpha_2\;(t)\;), \qquad (17)$$

According to (Bohm D. (1952)):

$$p = mv = \frac{\partial S}{\partial x} = m\frac{\partial x}{\partial t} = \hbar\ \alpha_1\ (t), \qquad (18)$$

The equation of movement is:

$$\frac{\partial x}{\partial t} = \frac{\hbar}{m}\ \left(b\ e^{-\frac{\Gamma t}{m}} + \frac{km}{\hbar\Gamma}\right),$$

$$x(t) = \frac{1}{m\Gamma}(b\ \ \hbar\ \ e^{-\Gamma t} + kt) + Cte. \qquad (19)$$

Results

Equation 19 allows to make the representation of a quantum system as the spatial configuration that evolves in time, as a trajectory under the action of the wave function, this is the main objective of the De Broglie-Bohm theory (or theory pilot wave (Avanzini,2016)). However, its standard formulation refers to the statistical set of its possible trajectories. The statistical set is introduced to establish the exact correspondence (Born's rule) between the probability density in the spatial configurations and the quantum distribution, that is, the squared modulus of the wave function. The pilot wave theory allows a formally self-consistent representation of quantum systems as the trajectory of a particle.

Equation 19 gives us the first result of the position of the particle as a function of time. Now from (Kostin 1972), where derive a Schrodinger equation for a Brownian particle interacting with a thermal environment. The function $W\ (t)$ of the equation (4) has the form:

$$W(t) = -\frac{\hbar\Gamma}{2i}\int \psi^*\ ln\left[\frac{\psi}{\psi^*}\right]\ \psi\ dx,$$

$$W(t) = -\hbar\ \Gamma\ F^2 \int G^2\ \ (\theta\ (t)\ + \alpha_1\ (t)\ x\ + \alpha_2\ (t)\ )\ \ dx\ . \qquad (20)$$

The integrals raised in equation 20 have as a solution

$$\int_0^{\infty} G(x)^2\ \ dx\ = \frac{c^2\hbar}{2am}\ ; \qquad \int_0^{\infty} G(x)^2\ x\ dx\ = \frac{c^2\hbar^2}{4a^2m^2},$$

Without considering the interpretation of Bohm (16), we find the equation of time evolution of the expected value of the position $x(t)$,

$$\langle x(t) \rangle = \int |\psi(x,t)|^2 \; x \; dx,$$

$$\langle x(t) \rangle = \frac{c^2 \hbar^2}{4a^2 m^2} e^{\frac{a}{g}\left(b\, e^{-gt} + \frac{kt}{m}\right)}, \qquad (21)$$

Discussion

In this paper the authors find a solution to the Schrödinger-Langevin equation, which includes terms of friction, is an article in which a linear exponential solution is proposed in , according to the expected behavior in the movement equation.

The solution obtained in the equation (19) contains a series of constants that must be implemented according to the initial conditions that are given according to the experimental set-up (Torres del Castillo G.F. 2016) Analogously, the solution obtained in the equation (21) which has a series of constants associated, which does not allow directly compare the two solutions.

Generally, in the literature there are studies of the behavior of the harmonic oscillator, in addition to the oscillator in an electric field, here we look for an expression in which only the contribution of an electric field is considered, since when considering the case of the oscillator a slip is obtained of energy levels.

Conclusions

The model used to solve the Schrödinger-Langevin equation, allowed to find the expected values of position and moment of a particle in a Coulomb potential, the temporal dependence of such solutions is made explicit, which allows finding the path equations of the particles.

Our results constitute an additional reason to formulate hydrodynamical versions of quantum evolution equations (Hatifi 2019). Although hydrodynamics is a natural language in this context, the mathematics is compatible with other theories of continuum mechanics, for example, elasticity (Holland 2005).

The trajectories of particles in a fluid can be considered analogous to full-wave ray theory in the limit of geometric optics. The last (classical) limit is obtained in circumstances where the internal potential and force are negligible compared to the external potential and force of the body, respectively. The result is a fluid dynamic representation of the classical Hamilton-Jacobi theory. It should be taken into account that this limit describes a continuous set of trajectories that do not interact moving in the potential V (as discussed in Holland 1996), instead of only one. The equation (21) allows to calculate possible quantum trajectories of the Broglie Bohmm type (Mariya 2020), as a classical limit, only the initial path conditions should be considered and the equation allows to determine the temporal evolution of the system.

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